Summary of Chap. 0.

1. Sets. Elements.

   Equality, Containment.

   Union, Intersection: Finite case, General Case.

   Complement. De Morgan Laws.

2. Ordered pairs. Cartesian Product. Relation.

   Function. Codomain. Domain. Image.

   injective (one-to-one), surjective (onto), bijective

   Inverse. Composition.

3. Well-ordering principle.

   Principle of Mathematical Induction. (first and second)

4. Equivalent Sets.

   Finite, Countably Infinite, Countable.

   Subsets, Finite Cartesian Products, Countable Unions of countable sets are countable.

   Uncountable Sets.

5. Least Upper Bound Property. (Axiom of Completeness)

   Archimedean Property. (Theorem 0.21)

   Density Theorem of $\mathbb{Q}$ (Theorem 0.22)

   Density Theorem of $\mathbb{R}\setminus\mathbb{Q}$ (Theorem 0.24)

   Absolute Value. Properties (Theorem 0.25)

   Triangle Inequalities.

Summary of Chap 1

1. Sequence. Limit of Sequence. Convergent Sequence.
   Neighborhood of a number. Expression of limit.
   Uniqueness of limit.
   Convergent Sequences are bounded (cut-off trick)

2. Cauchy Sequence.
   Cauchy Criterion: Every Cauchy sequence is convergent.
   Accumulation Point.
   Bolzano - Weierstrass Theorem: Every bounded infinite set of real numbers has at least one accumulation point.

3. Algebraic Limit Theorem:

$$\{a_n\} \to A,\ \{b_n\} \to B \Rightarrow \{a_n+b_n\} \to A+B. \quad \{a_n b_n\} \longrightarrow AB.$$

$$\{a_n\} \to A,\ \{b_n\} \to B \neq 0 \Longrightarrow \left\{\frac{a_n}{b_n}\right\} \to \frac{A}{B}.$$

   Order Limit Theorem.

$$\{a_n\} \to A,\ \{b_n\} \to B,\ a_n \le b_n,\ \forall n \Rightarrow A \le B$$

   Theorem 1.13:

$$\{a_n\} \to 0,\ \{b_n\} \text{ bounded} \Rightarrow \{a_n b_n\} \to 0.$$

4. Subsequence

$$\{a_n\} \text{ converges} \Leftrightarrow \forall \text{ subsequence } \{a_{n_k}\} \text{ converges.}$$

$$\{x_n\} \text{ bounded. } \forall \text{ subsequence } \{x_{n_k}\} \to x \Rightarrow \{x_n\} \to x.$$

   Increasing Sequence. Decreasing Sequence. Monotone Sequence.
   Monotone Convergence Theorem:

$$\{a_n\} \text{ monotone, then } \{a_n\} \text{ converges} \Leftrightarrow \{a_n\} \text{ is bounded}$$

Theorem 1.17:

$E \subseteq \mathbb{R}$, x_0 accumulation point of $E \Leftrightarrow \exists$ sequence $\{x_n\}$ in E, $x_n \to x_0$

Chap. 2 Stuff

1. Def of limit for functions: $f: D \to \mathbb{R}$, x_0 accumulation pt of D

"$\lim_{x \to x_0} f(x) = L$" means

$\forall \varepsilon > 0, \exists \delta > 0$, s.t $\forall x \in (x_0 - \delta, x_0 + \delta)$, $x \neq x_0$

$|f(x) - L| < \varepsilon$

Rmk: We don't care what $f(x_0)$ is at all.

The limit describes the behavior of f NEAR $x = x_0$.

"$\lim_{x \to x_0} f(x) \neq L$" means

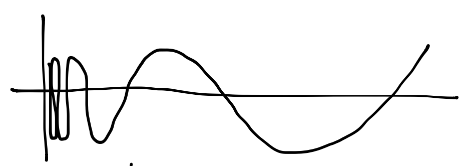
$\exists \varepsilon > 0, \forall \delta > 0$, $\exists x \in (x_0 - \delta, x_0 + \delta)$, $x \neq x_0$,

$|f(x) - L| \geq \varepsilon$

Example 2.3 $f: (0,1) \to \mathbb{R}$ $f(x) = \sin \frac{1}{x}$.

$x = 0$ is not a point in the domain but we can still talk about the behavior near $x = 0$.

$\lim_{x \to 0} f(x)$ DNE.



Since one notices $f\left(\frac{1}{n\pi}\right) = \sin \frac{1}{1/n\pi} = \sin n\pi = 0$

$f\left(\frac{1}{2n\pi + \pi/2}\right) = \sin\left(2n\pi + \frac{\pi}{2}\right) = 1$

$\lim_{x \to 0} f(x)$ cannot be 0, since

for $\varepsilon = \frac{1}{2}$, $\forall \delta > 0$, $\exists x = \frac{1}{2n\pi + \pi/2} < \delta$. $|f(x) - 0| = 1 \geq \frac{1}{2}$

$\lim_{x\to 0} f(x)$ cannot be any number $L \neq 0$ since

$\forall L > 0$, take $\varepsilon = \frac{1}{2}L$, then.

$\forall \delta > 0$, $\exists\, x = \frac{1}{n\pi} < \delta$, s.t. $|f(x) - L| = |0 - L|$

$= L > \frac{1}{2}L = \varepsilon.$

Exercise: $L < 0$ case.

Exercise 6: Hint: Note that if $x = \frac{1}{2n\pi}$ then $f(\frac{1}{2n\pi}) = 1$

$x = \frac{1}{n\pi + \frac{\pi}{2}}$, $f\left(\frac{1}{n\pi+\frac{\pi}{2}}\right) = 0.$

The argument above carries over trivially.

2. Connection to sequential limits

Theorem: (   ) $f: D \to \mathbb{R}$. x_0 accumulation point of D.

Then $\lim_{x\to x_0} f(x)$ exists $\Leftrightarrow$ $\forall \{x_n\}_{n=1}^{\infty}$ in D that converges to x_0 with $x_n \neq x_0$, $\lim_{n\to\infty} f(x_n)$ exists.

Rmk: We can use sequential limits to describe limit of functions.

Theorem: if any $\{x_n\}_{n=1}^{\infty}$ in D, $x_n \to x_0$, $x_n \neq x_0$ makes $\{f(x_n)\}_{n=1}^{\infty}$ a Cauchy sequence, then $\lim_{n\to\infty} f(x_n)$ exists.

3. Theorem: $\lim_{x\to x_0} f(x)$ exists $\Rightarrow$ $f(x)$ is bounded near x_0

i.e. $\exists\, M \geq 0$, $\exists\, \delta > 0$,

$\forall x \in (x_0 - \delta, x_0 + \delta) \cap D$, $|f(x)| \leq M$

4. Algebraic Limit Theorem:

If $\lim_{x\to x_0} f(x) = A$, $\lim_{x\to x_0} g(x) = B$

$$\Rightarrow \lim_{x\to x_0} \left(f(x)+g(x)\right) = A+B$$

$$\lim_{x\to x_0} f(x)g(x) = AB$$

If in addition, $\lim_{x\to x_0} g(x) = B \neq 0$,

$$\Rightarrow \lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{A}{B}.$$

5. Order limit theorem.

$f(x) \leq g(x)$, $\forall x \in D$. $\lim_{x\to x_0} f(x)$, $\lim_{x\to x_0} g(x)$ exists

$$\Rightarrow \lim_{x\to x_0} f(x) \leq \lim_{x\to x_0} g(x).$$

6. <u>Theorem</u>: $\lim_{x\to x_0} f(x) = 0$ and $g(x)$ is bounded near x_0

$\Rightarrow \lim_{x\to x_0} f(x)g(x)$ exists and equal to zero.

<u>Example</u>: $f(x)=x^2$: $\mathbb{R}\to\mathbb{R}$ satisfies $\lim_{x\to x_0} f(x) = x_0^2$.

$$|f(x)-x_0^2| = |x^2-x_0^2| = |x+x_0|\cdot|x-x_0|$$

Want: $\forall \varepsilon>0$, $\exists \delta>0$, $\forall x\in(x_0-\delta, x_0+\delta)$, $x\neq x_0$, $|f(x)-x_0^2|<\varepsilon$

i.e. find δ, s.t. $|x+x_0|\cdot|x-x_0| < \varepsilon$, $\forall x_0 \neq x \in (x_0-\delta, x_0+\delta)$

WLOG, assume $x_0>0$. Take $\delta = \min\left(x_0, \frac{\varepsilon}{3x_0}\right)$. $|x-x_0|<\delta \Rightarrow |x+x_0|<3x_0$.

So $|f(x)-x_0^2| = |x+x_0|\,|x-x_0|$

$$< 3x_0\cdot\frac{\varepsilon}{3x_0} = \varepsilon$$

<u>Ex</u>: $x_0<0$.